International Conference on Flow Physics and its Simulation
- In memory of Prof. Jaw Yen Yang
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In memory of Prof. Jaw Yen Yang

What we have done with Godunov scheme in the past ?

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Outline

- Some comments on Godunov type finite volume method
- Boundary Variation Diminishing (BVD)

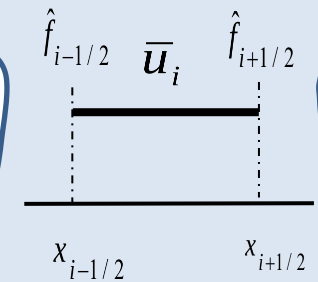
  – a new guideline to construct high-fidelity Godunov schemes
- Implementation
- Summary

Godunov type finite volume method

Finite volume formulation for conservation law

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x} \quad \Rightarrow \quad \frac{d\bar{u}_i}{dt} = -\frac{1}{\Delta x}\left(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}\right)$$

$$\bar{u}_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\,dx$$



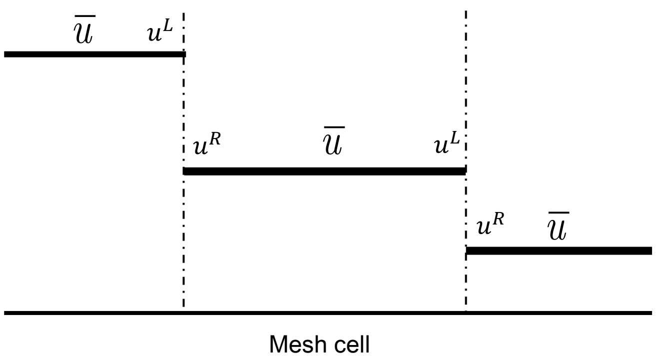
Godunov (1959)

Godunov type finite volume method

The key stone for modern high-resolution schemes

(1) Reconstruction :

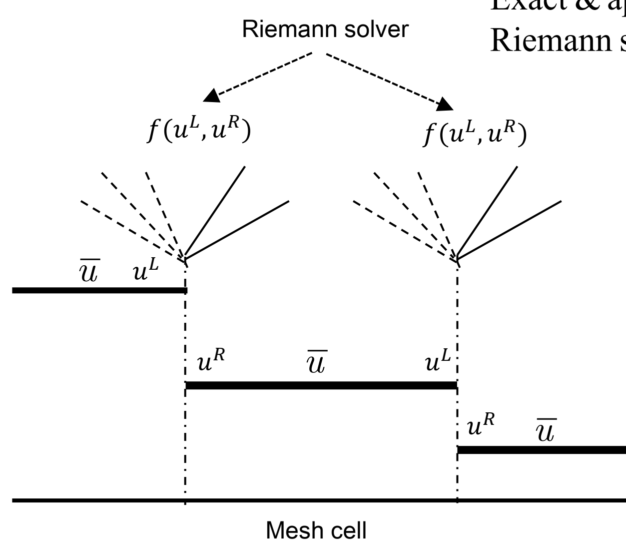
Given cell - average $\bar{u}_i$, find left - side u^L and right - side values u^R at cell boundaries



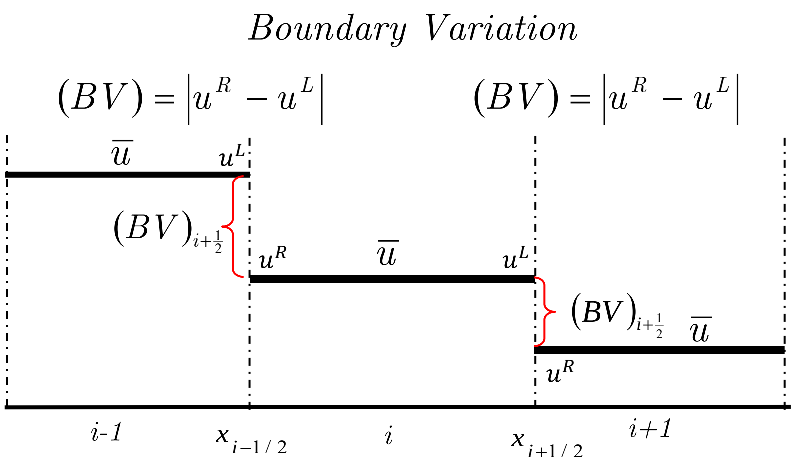
(2) Evolution :

Get the numerical fluxes $\hat{f}_{i+\frac{1}{2}} = f\left(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}}\right)$ by Riemann solvers

Exact & approximate Riemann solvers



Godunov type finite volume method



Observation 1 :

The boundary variation (jump) (BV) is a natural and inevitable product of the FVM formulation

Observation 2 :

The Riemann solvers can be cast in a canonical form

$$\hat{f}_{i+\frac{1}{2}}=\frac{1}{2}\left(f(u^L_{i+\frac{1}{2}})+f(u^R_{i+\frac{1}{2}})\right)-\frac{1}{2}\left|\alpha_{i+\frac{1}{2}}\right|\left(u^R_{i+\frac{1}{2}}-u^L_{i+\frac{1}{2}}\right)$$

↘ Central scheme

↘ Effective dissipation

$$\mapsto \frac{1}{2}\left|\alpha_{i+\frac{1}{2}}\right|(BV)_{i+\frac{1}{2}}$$

Observation 2 suggests that minimizing BV is the key to reduce numerical dissipation

Godunov type finite volume method

Observations 3 :

Historicaly, practices to develop high - order Godunov finite volume methods stick to the assumptions

◊ the jumps always occur at cell boundaries

◊ the profile within a cell is smooth

$\Rightarrow$ seek high - order polynomial reconstructions within each cell

Prof. J..Yang
Important contributions from 1980s to 2010s

Representative works :

Single - moment type (one DOF for one cell) :

$\Rightarrow$ piece - wise constant $\rightarrow$ linear(MUSCL) $\rightarrow$ parabolic (PPM) $\rightarrow$ ENO $\rightarrow$ WENO

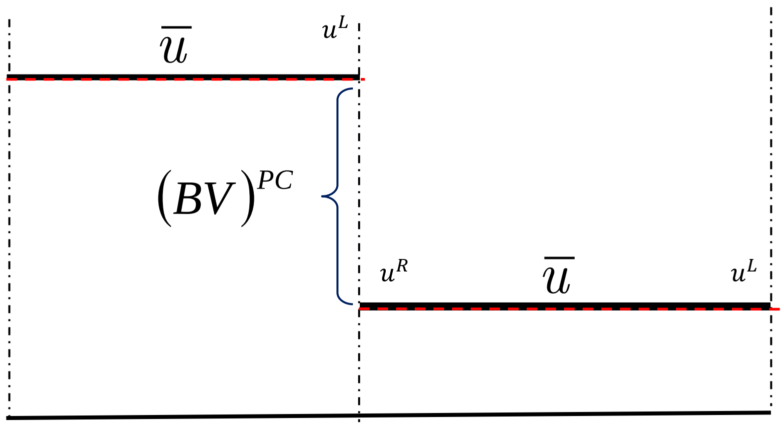
Multi - moment type (multiple DOFs for one cell) :

$\Rightarrow$ spectral element, discontinuous Galerkin (DG), flux reconstruction (FR/CPR)

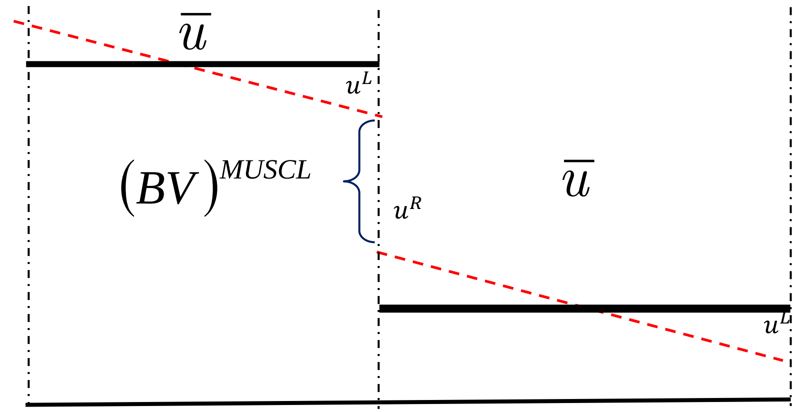
Higher-order reconstructions

$$\text{Boundary Variation} \quad (BV) = \left| u^R - u^L \right|$$

Piece-wise constant

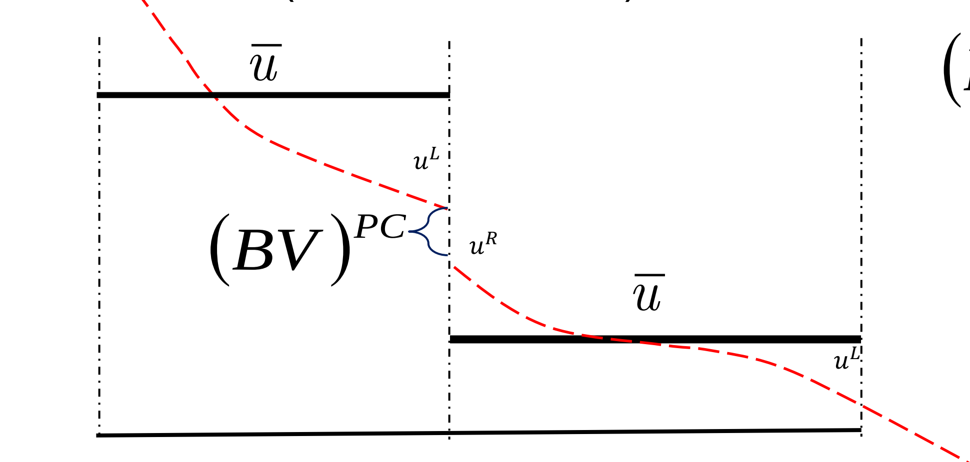


MUSCL



High-order polynomial

(ENO,WENO,…)

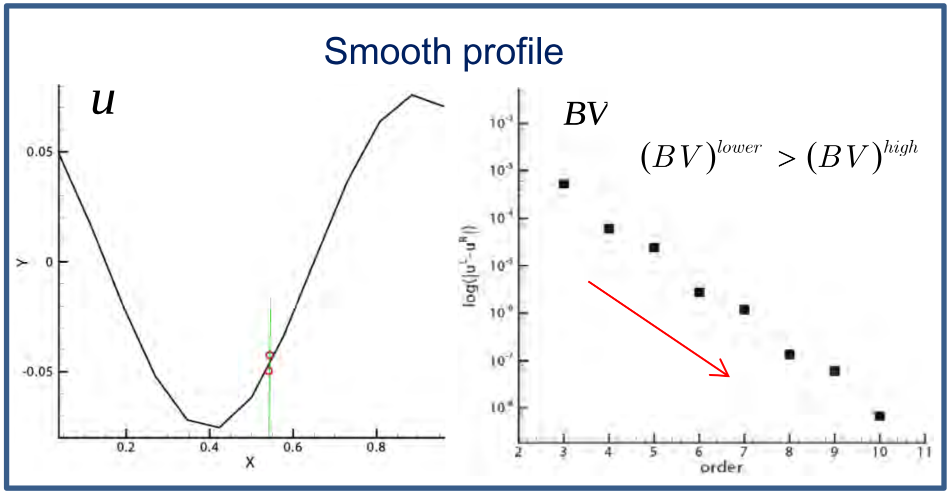


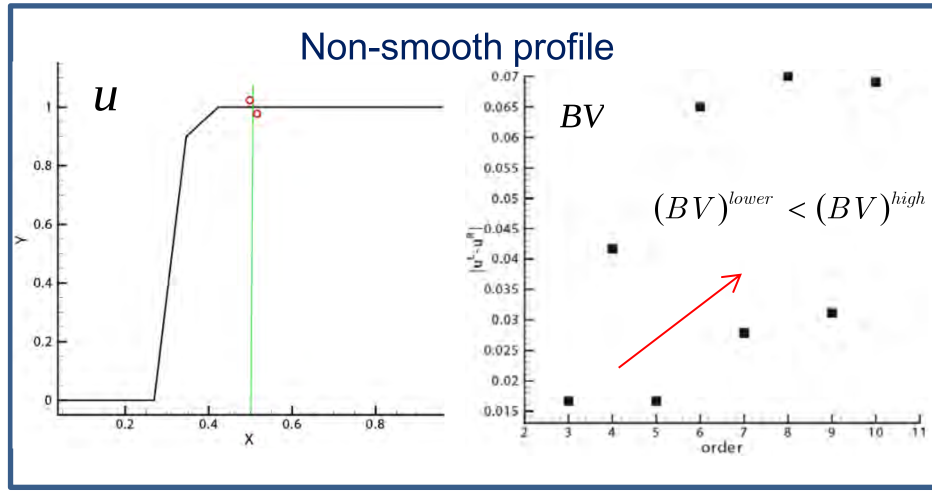
$$(BV)^{HOP} < (BV)^{MUSCL} < (BV)^{PC}$$

Does higher-order reconstruction always generate smaller BV?

What we have done in the past really lead to reconstructions that minimize BV ?

A numerical experiment





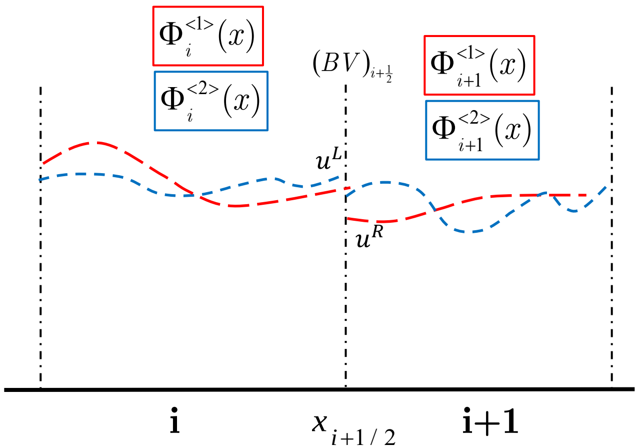
- YES ! for smooth solution (*Weierstrass approx. Theorem (1885)*) ✓
- Very questionable for discontinuous solution ??

We may have ignored BV as a key factor in reconstructions

A new strategy:
design reconstructions that minimize boundary jumps
→ BVD (Boundary Variation diminishing) principle

Boundary Variation Diminishing (BVD)

BVD: a new practical guideline for reconstruction that adaptively chooses proper interpolation functions so as to minimize the jumps between the left- and right- side values, u^L and u^R, at cell interface.



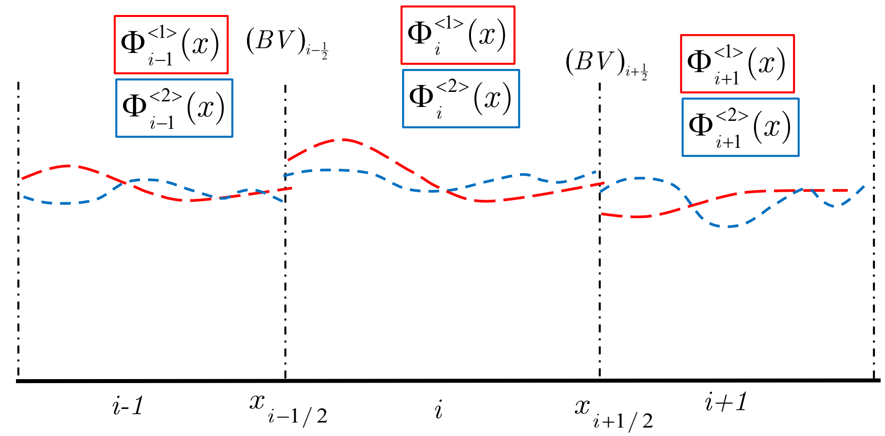
Remarks:

- The BVD algorithm reduces the reconstructed jumps at cell interfaces and thus the numerical dissipation in Riemann solvers, which leads to substantial improvements in numerical solutions.
- A general and reliable principle to design high-resolution schemes for both smooth and non-smooth solutions.
- For smooth solution, the BVD reconstruction naturally realizes the highest possible interpolation.
- For discontinuous solution, the BVD algorithm prefers other reconstructions rather than polynomials.

Implementing BVD principle

BVD algorithm 1 (a simple version for structured mesh) ✔

1. Prepare two (or more) cell-wise interpolation functions for the reconstructed variable $u(x)$,
   $\Phi_i^{<1>}(x)$: higher-order less monotonic
   $\Phi_i^{<2>}(x)$: lower-order more monotonic



2. Find $\Phi_i^{<p>}(x)$ and $\Phi_{i+1}^{<q>}(x)$, $p, q = 1, 2$, so that the boundary variation (BV)
   $BV(\Phi)_{i+\frac{1}{2}} = |\Phi_i^{<p>}(x_{i+\frac{1}{2}}) - \Phi_{i+1}^{<q>}(x_{i+\frac{1}{2}})|$
   is minimized;

3. If a different choice, $\Phi_i^{<p'>}(x)$, for cell i is made when applying step 2 to the neighboring interface $x_{i-\frac{1}{2}}$ to minimize
   $BV(\Phi)_{i-\frac{1}{2}} = |\Phi_{i-1}^{<p'>}(x_{i-\frac{1}{2}}) - \Phi_i^{<q'>}(x_{i-\frac{1}{2}})|$
   I.e. $\Phi_i^{<p>}(x) \neq \Phi_i^{<p'>}(x)$,

$$\Phi_i^{<p>}(x) = \begin{cases} \Phi_i^{<1>}(x), \text{ if } \left(\Phi_i^{<p>}(x_{i+\frac{1}{2}}) - \Phi_{i+1}^{<q>}(x_{i+\frac{1}{2}})\right)\left(\Phi_{i-1}^{<p'>}(x_{i-\frac{1}{2}}) - \Phi_i^{<q'>}(x_{i-\frac{1}{2}})\right) < 0, \\ \Phi_i^{<2>}(x), \text{ otherwise.} \end{cases}$$

$$\text{or } \quad \Phi_i^{<p>}(x) = \begin{cases} \Phi_i^{<1>}(x), \text{ if } (\bar{u}_i - \bar{u}_{i+1})(\bar{u}_{i-1} - \bar{u}_i) < 0, \\ \Phi_i^{<2>}(x), \text{ otherwise.} \end{cases}$$

Implementing BVD principle

BVD algorithm 2 (least square approach)

1. Prepare a cell-wise interpolation function over mesh cell Ω_i
   $\tilde{\Phi}_i(x) = \omega_i \Phi_i^{<1>}(x) + (1-\omega_i)\Phi_i^{<2>}(x)$, with $0 \leq \omega_i \leq 1$ and
   $\Phi_i^{<1>}(x)$: higher-order less monotonic
   $\Phi_i^{<2>}(x)$: lower-order more monotonic

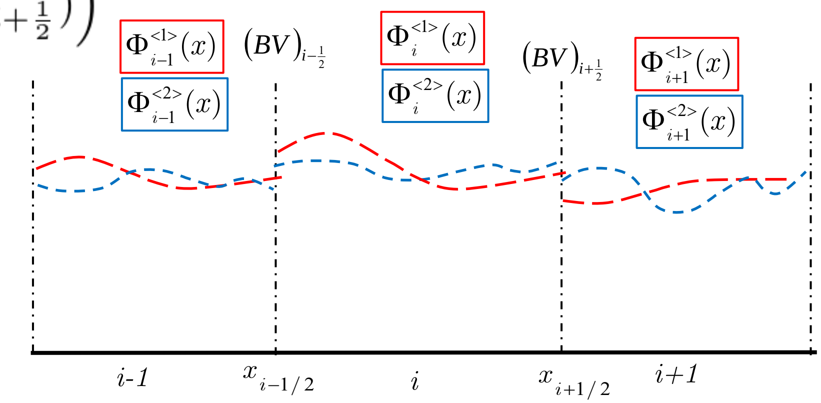
2. Find ω_i, so that the total variations on all cell boundaries

$$TBV_{i+\frac{1}{2}} = \sum_{i=1}^{I} \left(\tilde{\Phi}_i(x_{i+\frac{1}{2}}) - \tilde{\Phi}_{i+1}(x_{i+\frac{1}{2}}) \right)^2$$

   is minimized. A least-square method leads

$$\frac{\partial}{\partial \omega_i}(TBV_i) = 0,$$

   a simultanously liked linear system.



3. Once ω_i is determined, $\tilde{\Phi}_i(x)$ is used for cell Ω_i to compute numerical fluxes at cell boundaries.

Some BVD admissible reconstructions

- $\Phi_i^{<1>}(x)$:
  5th-order WENO type schemes
  $$\Phi_i^{<1>}(x_{i\pm\frac{1}{2}}) = \omega_0 u^{(0)}_{i\pm\frac{1}{2}} + \omega_1 u^{(1)}_{i\pm\frac{1}{2}} + \omega_2 u^{(2)}_{i\pm\frac{1}{2}}, \qquad \text{Jiang \& Shu (1996)}$$
  where
  $$u^{(0)}_{i\pm\frac{1}{2}} = \frac{1}{3}\overline{u}_{i\mp2} - \frac{7}{6}\overline{u}_{i\mp1} + \frac{11}{6}\overline{u}_i,\ u^{(1)}_{i\pm\frac{1}{2}} = -\frac{1}{6}\overline{u}_{i\mp1} + \frac{5}{6}\overline{u}_i + \frac{1}{3}\overline{u}_{i\pm1},$$
  $$u^{(2)}_{i\pm\frac{1}{2}} = \frac{1}{3}\overline{u}_i + \frac{5}{6}\overline{u}_{i\pm1} - \frac{1}{6}\overline{u}_{i\pm2}.$$

  - WENO-Z scheme (Borges et al, 2008),
    $$\omega_k = \frac{\alpha_k}{\sum_{k=0}^{2}\alpha_k},\ \alpha_k = d_k\left(1 + \left(\frac{\tau_5}{\beta_k+\epsilon}\right)^q\right),\ \tau_5 = |\beta_2 - \beta_0|,\ k = 0,1,2,$$
    The smoothness indicator Jiang & Shu (1996)
    $$\beta_j = \sum_{l=1}^{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Delta x^{2l-1}\left(\frac{d^l u^{(j)}(x)}{dx^l}\right)^2 dx, \quad j = 0,1,2.$$
  - TENO5 scheme (Fu, 2016),
    $$\omega_k = \frac{d_k\delta_k}{\sum_{k=0}^{2} d_k\delta_k},\ k = 0,1,2;\ \delta_k = \begin{cases} 0, & \text{if } \chi_k < C_T \\ 1, & \text{otherwise} \end{cases}\ k = 0,1,2,\ \ C_T = 10^{-5}$$
    The smoothness measure
    $$\chi_k = \frac{\gamma_k}{\sum_{k=0}^{2}\gamma_k},\ k = 0,1,2,\ \ \gamma_k = \left(C + \frac{\tau_k}{\beta_k+\epsilon}\right)^q,\ k = 0,\ldots,K-3,$$
    where $K = 5$, $C = 1$ and $q = 6$ are suggested.

Some BVD admissible reconstructions

- $\Phi_i^{<2>}(x)$:
  THINC scheme (Xiao et al, 2005, 2012, ...)

$$\Phi_i^{<2>}(x) = u_{min} + \frac{u_{max}}{2}\left(1 + \gamma \tanh\left(\beta\left(\frac{x - x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \tilde{x}_i\right)\right)\right),$$

where $u_{min} = \min(\bar{u}_{i-1}, \bar{u}_{i+1})$, $u_{max} = \max(\bar{u}_{i-1}, \bar{u}_{i+1}) - u_{min}$,
$\gamma = \operatorname{sgn}(\bar{u}_{i+1} - \bar{u}_{i-1})$, parameter β is used to control the jump thickness.

$$\Phi_i^{<2>}(x_{i+\frac{1}{2}}) = u_{min} + \frac{u_{max}}{2}\left(1 + \gamma \frac{\tanh(\beta) + A}{1 + A\tanh(\beta)}\right),$$

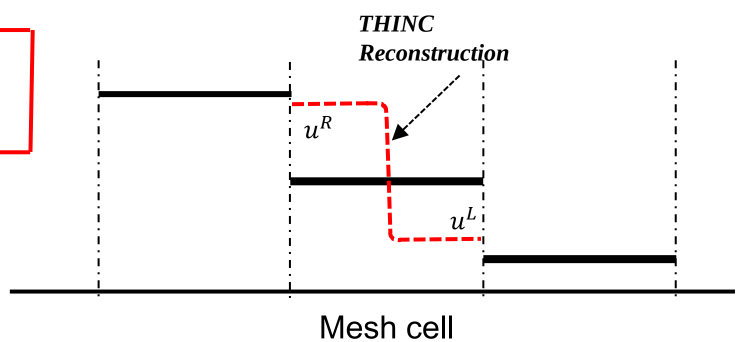
$$\Phi_i^{<2>}(x_{i-\frac{1}{2}}) = u_{min} + \frac{u_{max}}{2}(1 + \gamma A),$$

where $A = \frac{B/\cosh(\beta) - 1}{\tanh(\beta)}$, $B = \exp\left(\gamma\beta\left(2\frac{\bar{u}_i - \bar{u}_{min} + \epsilon}{u_{max} + \epsilon} - 1\right)\right)$ and $\epsilon = 10^{-20}$.

A Sigmoid function, exact monotonicity and minimum/maximum bounded.

$$\frac{d\Phi(x)}{dx} > 0 \quad \text{or} \quad \frac{d\Phi(x)}{dx} < 0$$

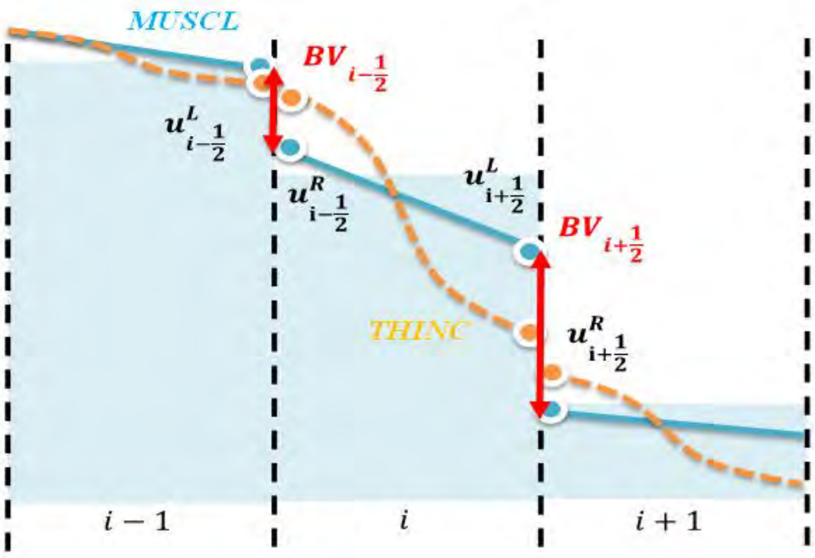
$$u_{min} \leq \Phi(x) \leq u_{max}$$

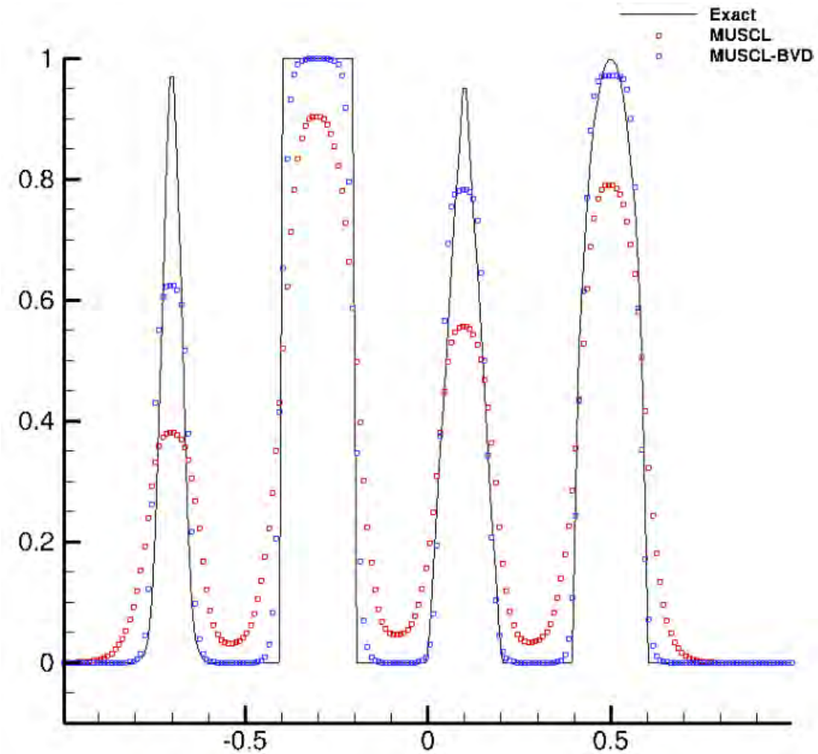


Represent jump within mesh cell
$\longrightarrow$ Reduce BV at cell boundary

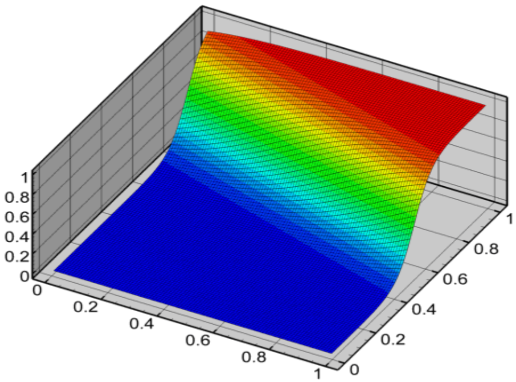
It works !

Jiang and Shu's advection test





THINC method with multi-dimensional reconstructions



Multi-dimensional hyperbolic tangent function

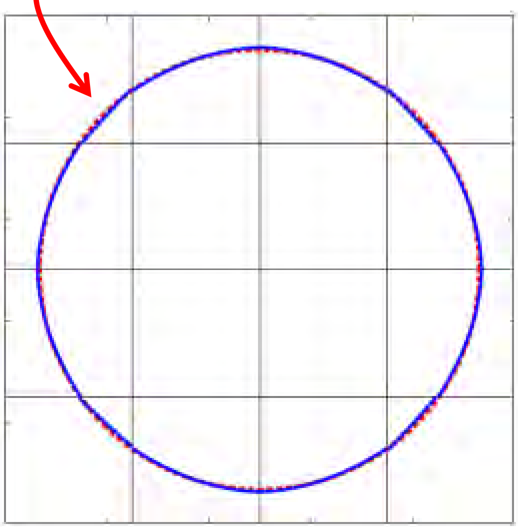
$$\hat{H}(\mathbf{X}) = \frac{1}{2}\left(1 + \tanh\left(\beta\left(P(\mathbf{X}) + d\right)\right)\right), \quad X, Y, Z \in [0, 1]$$

Surface function (linear)

$$P(\mathbf{X}) = n_X X + n_Y Y + n_Z Z$$

Surface function (quadratic)

$$P(\mathbf{X}) = c^{(X)} a_{200} X^2 + c^{(Y)} a_{020} Y^2 + c^{(Z)} a_{002} Z^2 + c^{(X)} c^{(Y)} a_{110} XY + c^{(Y)} c^{(Z)} a_{011} YZ + c^{(X)} c^{(Z)} a_{101} XZ + a_{100} X + a_{010} Y + a_{001} Z,$$



$l_Y, \quad \frac{\partial P}{\partial Z}(\mathbf{X}_c) = n_Z,$

$= l_{YY}, \quad \frac{\partial^2 P}{\partial Z^2}(\mathbf{X}_c) = l_{ZZ},$

$\mathbf{X}_c) = l_{YZ}, \quad \frac{\partial^2 P}{\partial Z \partial X}(\mathbf{X}_c) = l_{XZ}$

$, \quad (a, b = X, Y, Z)$ curvatu

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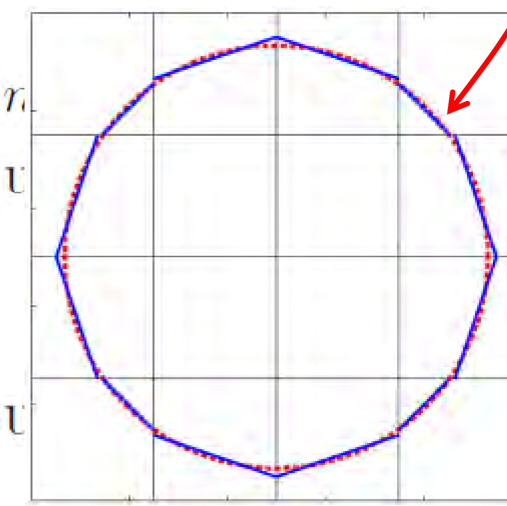
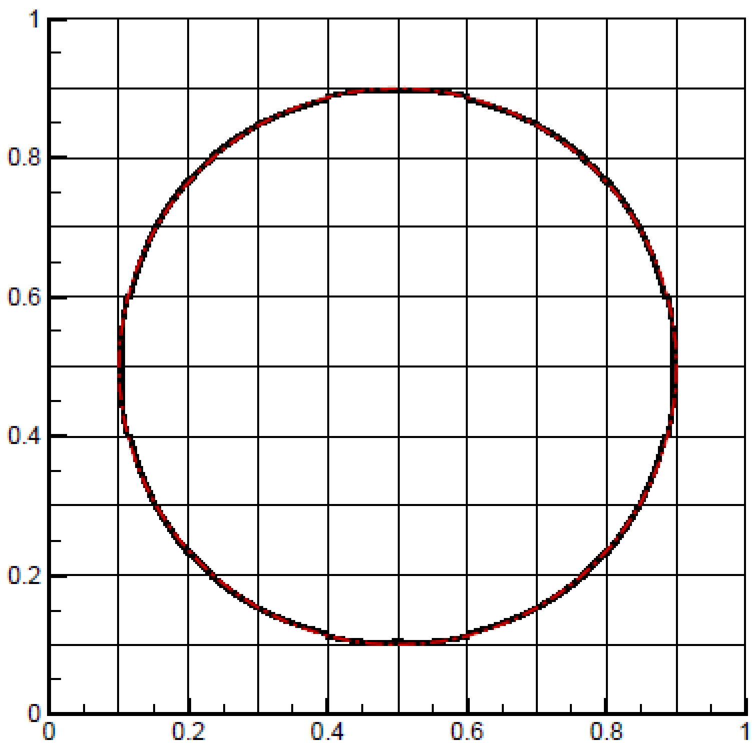
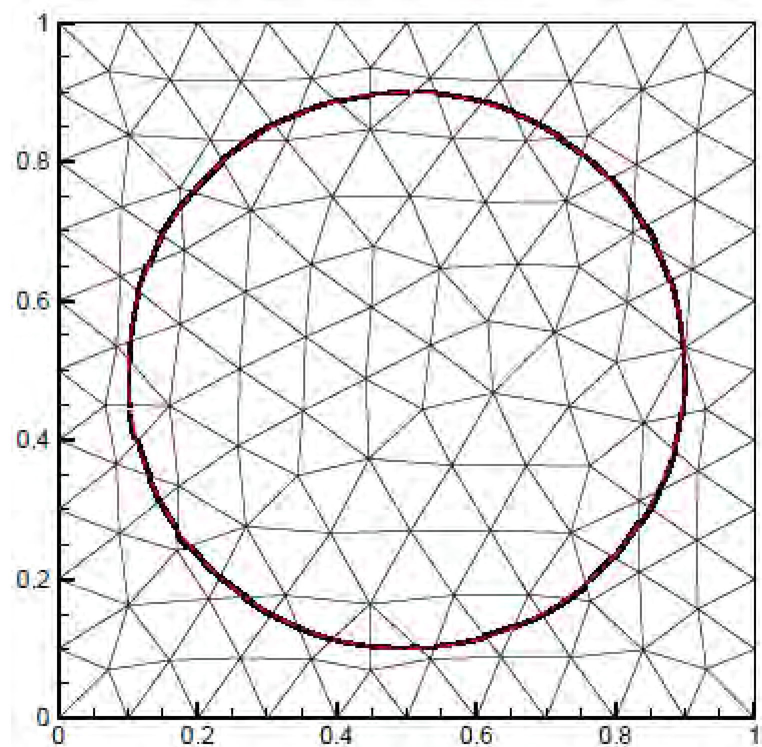


Reconstructions of a circular jump with quadratic interface representation

$$H(\mathbf{x}) = \begin{cases} 1, & \text{if } \sqrt{(x-0.5)^2 + (y-0.5)^2} \le \text{R}. \\ 0, & \text{otherwise.} \end{cases}$$





Numerical test

1D advection equation

$u_t + u_x = 0$

with initia smooth profile $u(x,0) = \sin(\pi x),\ x \in [-1,1]$

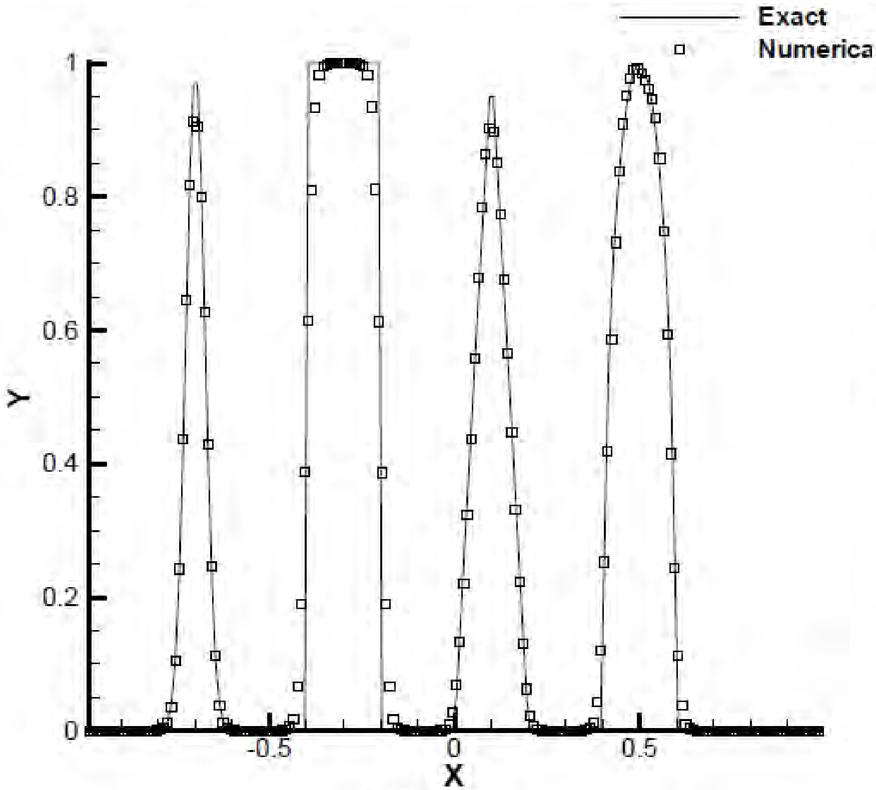
Errors and convergence rate of 1D advection equation, $t = 2.0$.

N	BVD-WENOZ-THINC scheme				WENO-Z scheme			
	L_1 error	Order	L_∞	Order	L_1 error	Order	L_∞	Order
20	2.14e-04		3.65e-06		2.14e-04		3.65e-06	
40	6.40e-06	5.07	1.03e-05	5.10	6.40e-06	5.07	1.03e-05	5.10
80	2.00e-07	5.00	3.18e-07	5.02	2.00e-07	5.00	3.18e-07	5.02
160	6.32e-09	4.99	9.96e-09	5.00	6.32e-09	4.99	9.96e-09	5.00
320	2.04e-10	4.00	3.20e-10	4.96	2.04e-10	4.00	3.20e-10	4.96

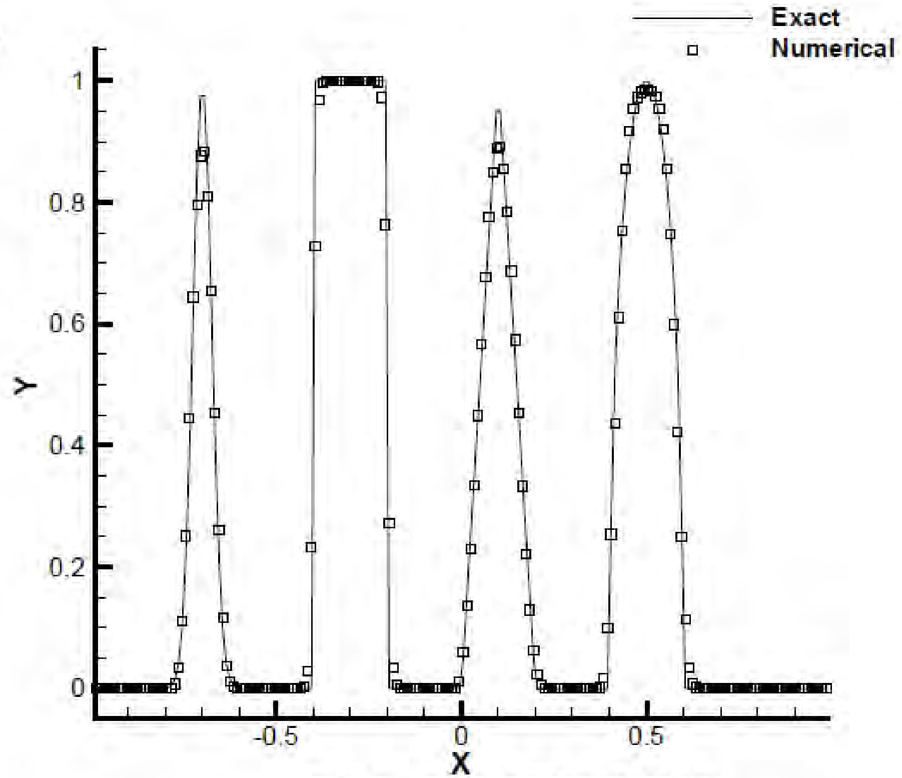
The BVD algorithm automatically choose the highest polynomial for smooth solution

Numerical test

Jiang and Shu's advection test



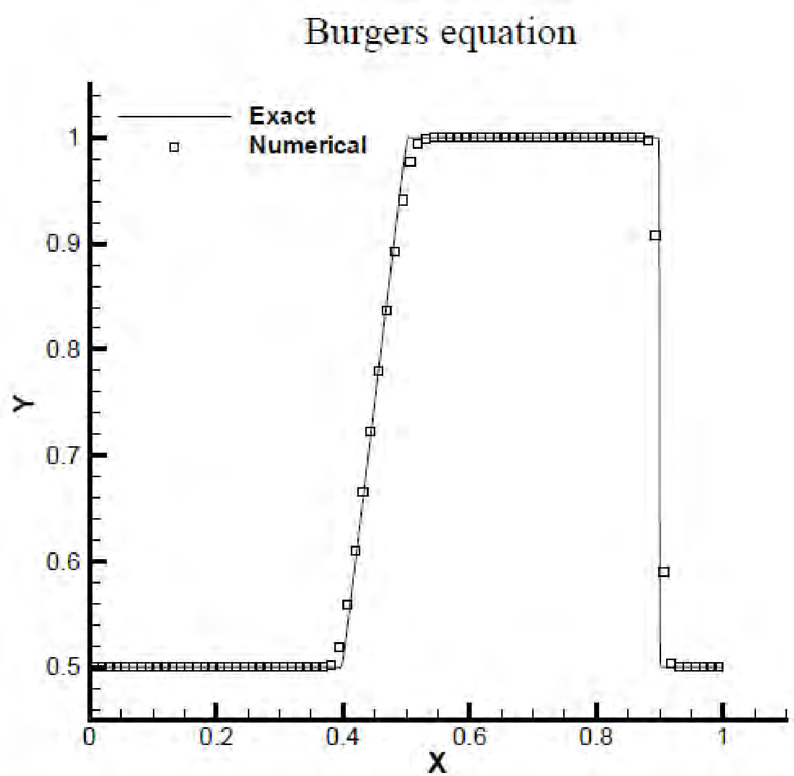
WENO-Z

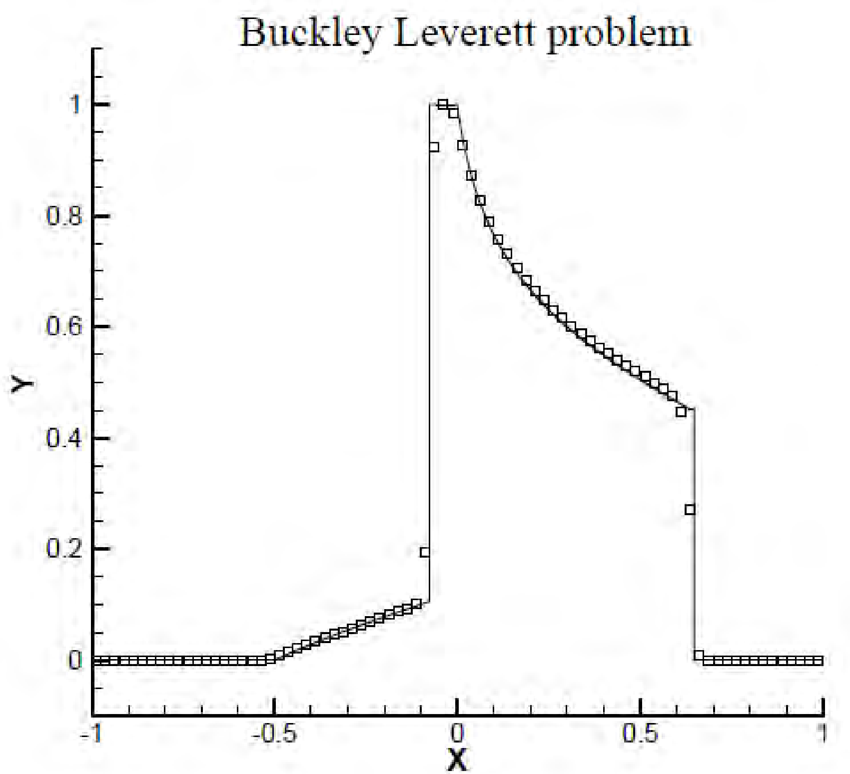


BVD- WENOZ-THINC

Numerical test

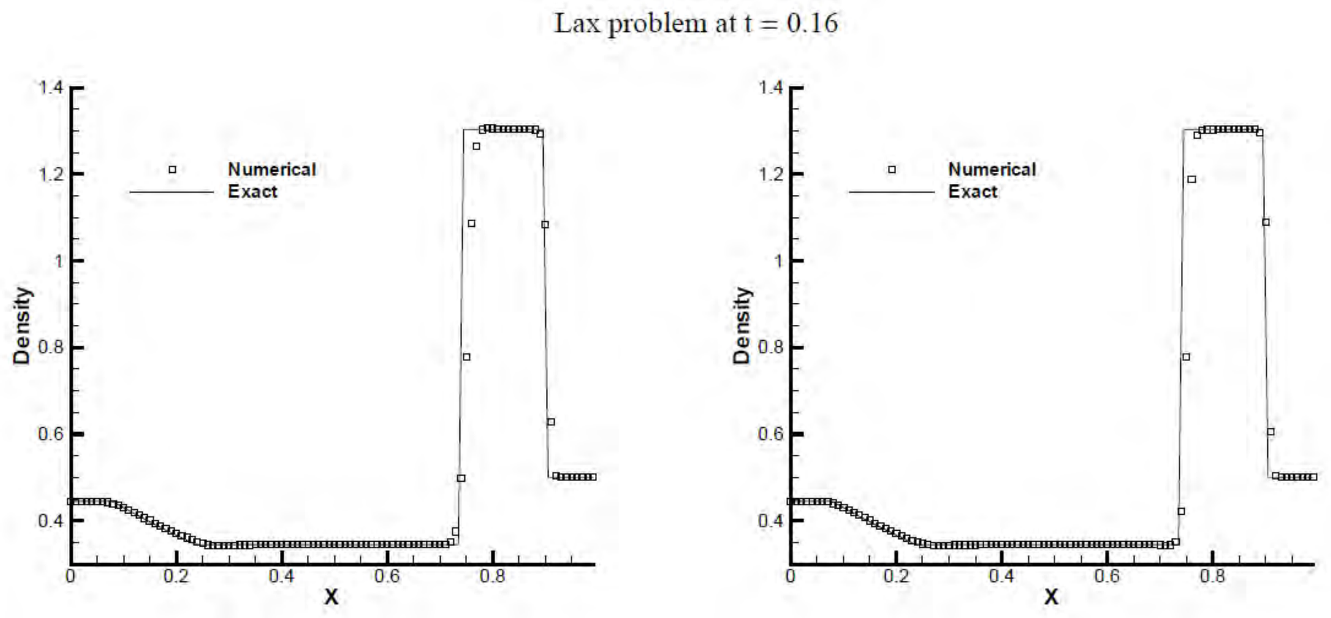
Nonlinear scalar hyperbolic conservation equations



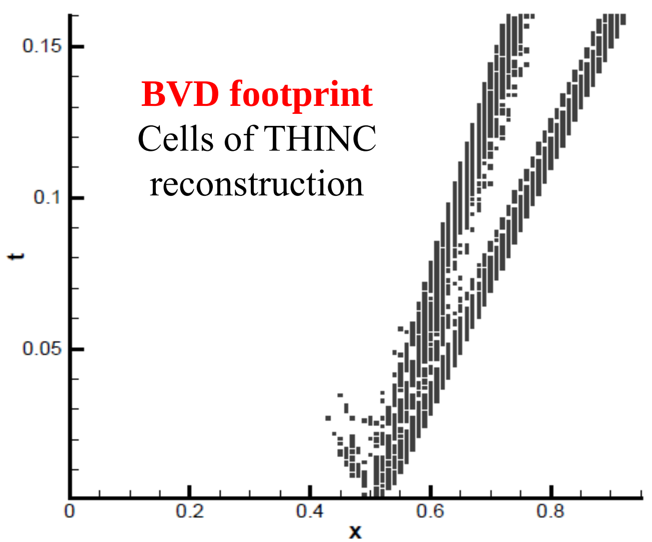


Numerical test

Lax Problem (Euler equation)



BVD serves well as a discontinuity detector

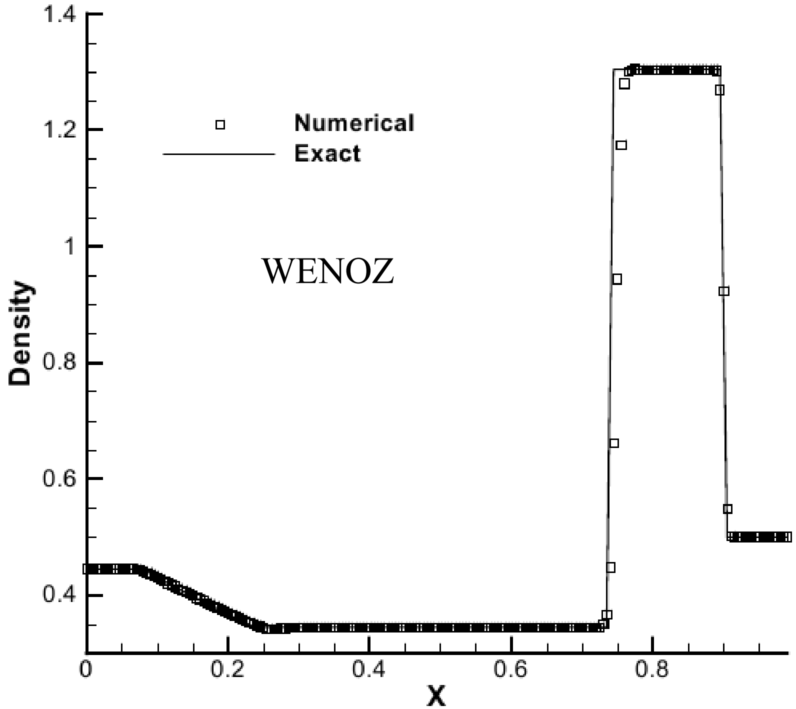


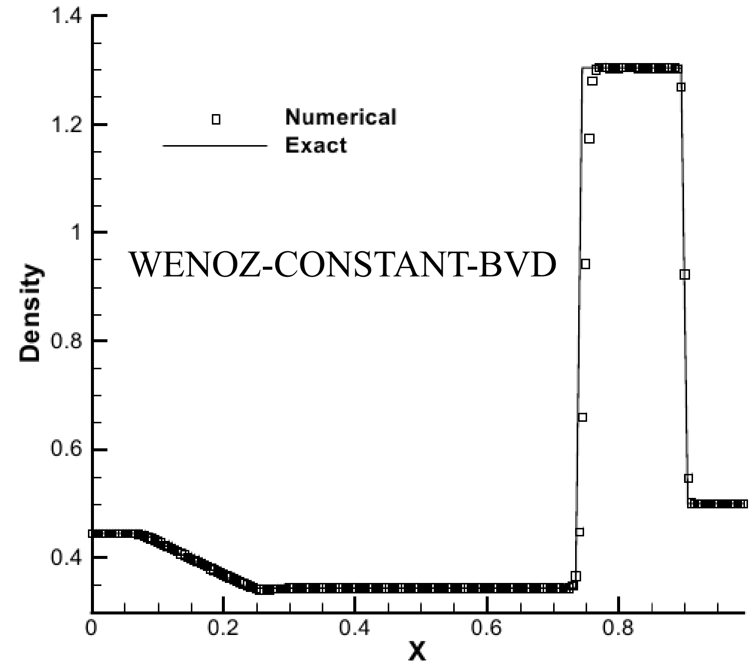
Extra examples

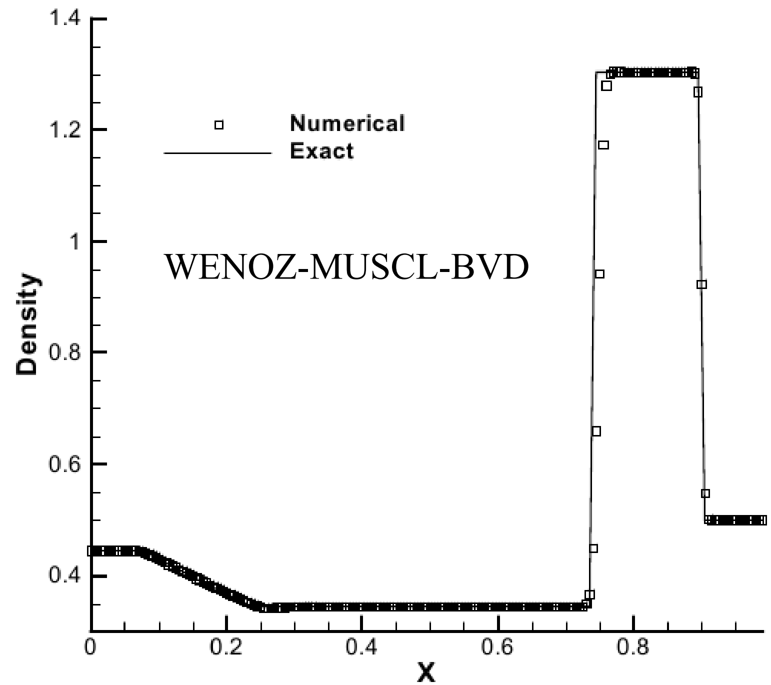
Lax Problem (Euler equation)200 cells

BVD using other combinations

(WENO + others)

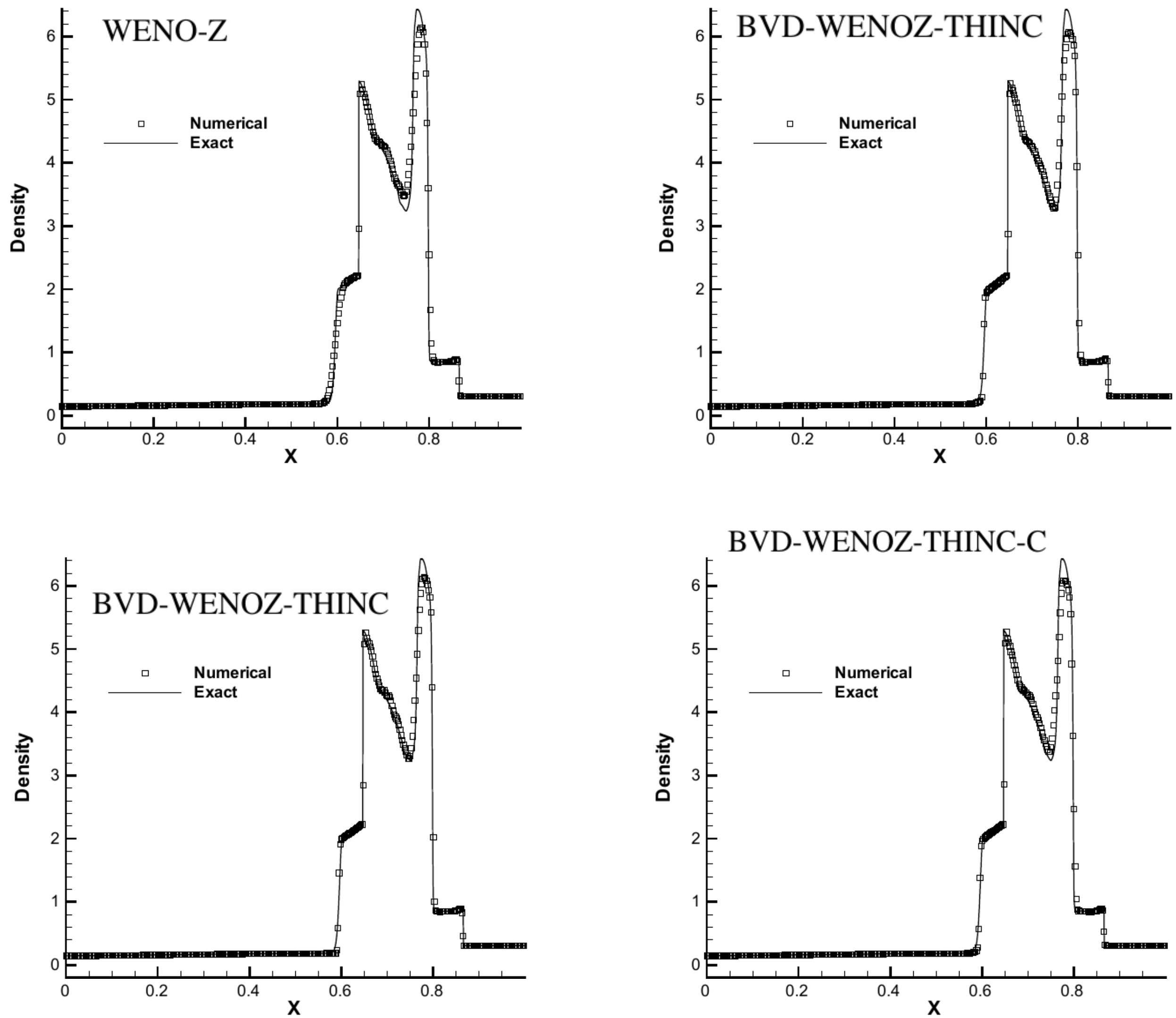






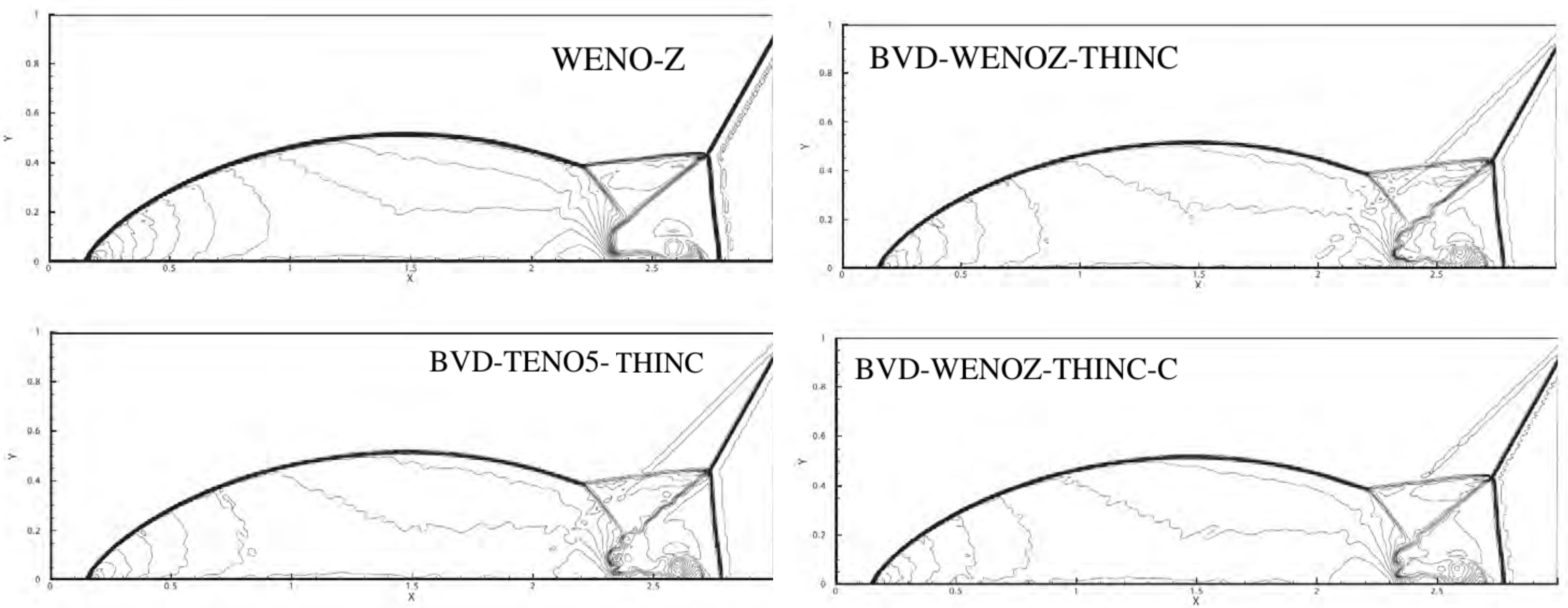
Numerical test

Two blast waves 400 cells at t=0.038



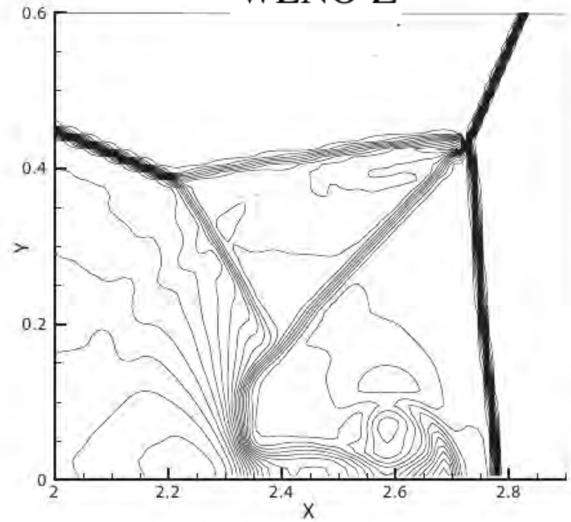
Numerical test

Double Mach reflection at t=0.2 on 100X320 mesh

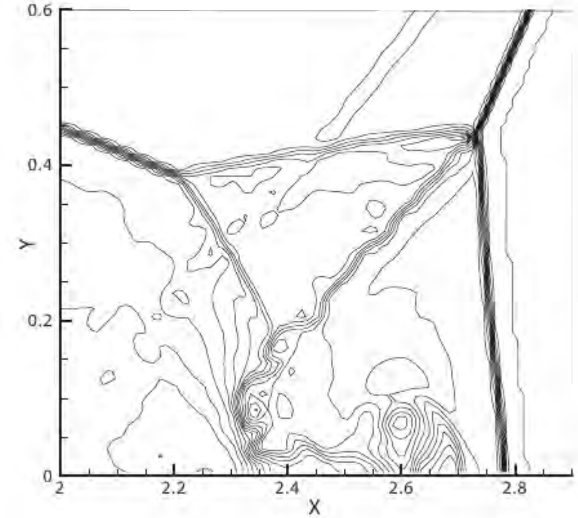


Double Mach reflection at t=0.2 on 100X320 mesh

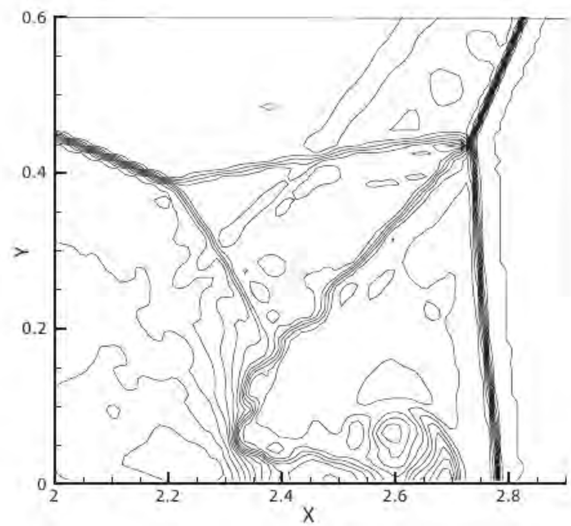
WENO-Z



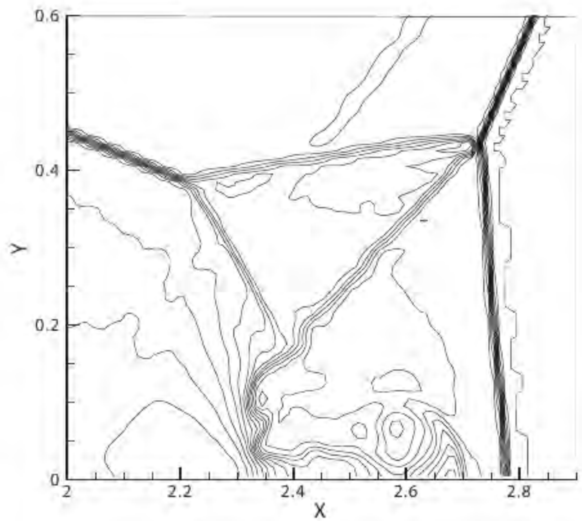
BVD-WENOZ-THINC



BVD-TENO5- THINC

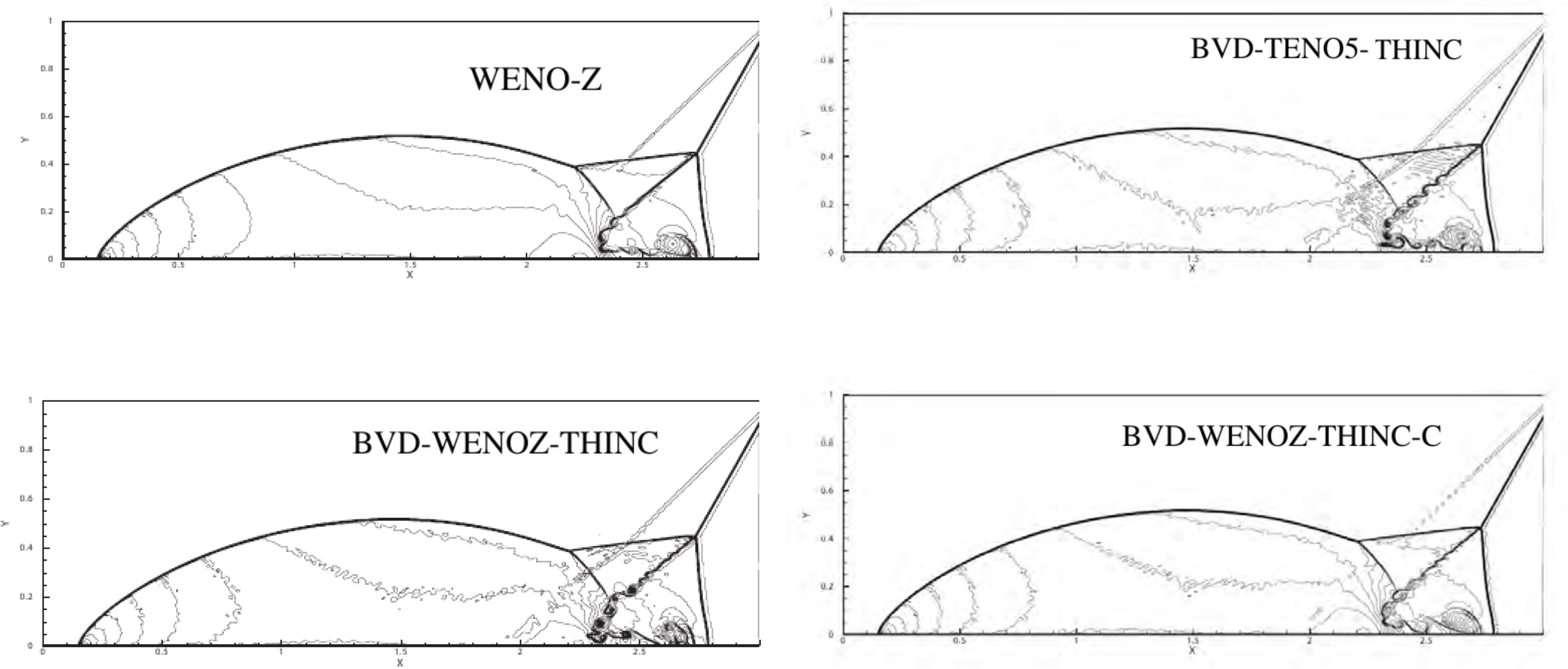


BVD-WENOZ-THINC-C



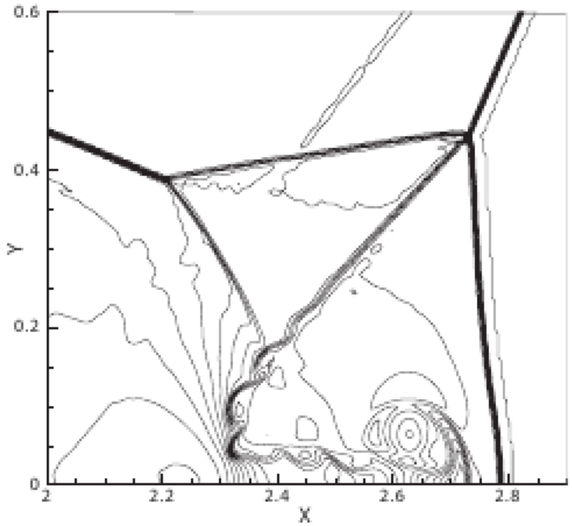
Numerical test

Double Mach reflection at t=0.2 on 200X640 mesh

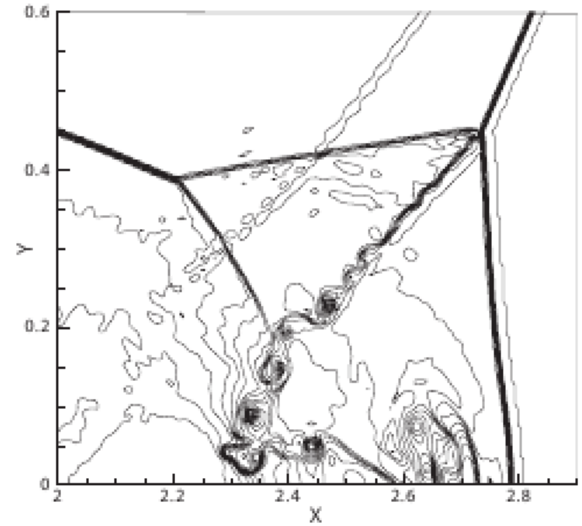


Double Mach reflection at t=0.2 on 200X640 mesh

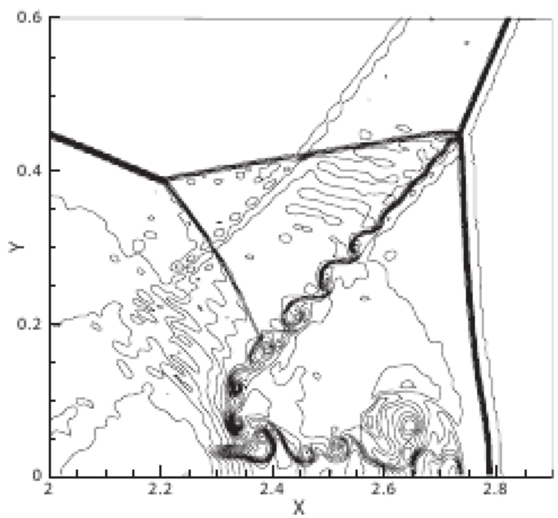
WENO-Z



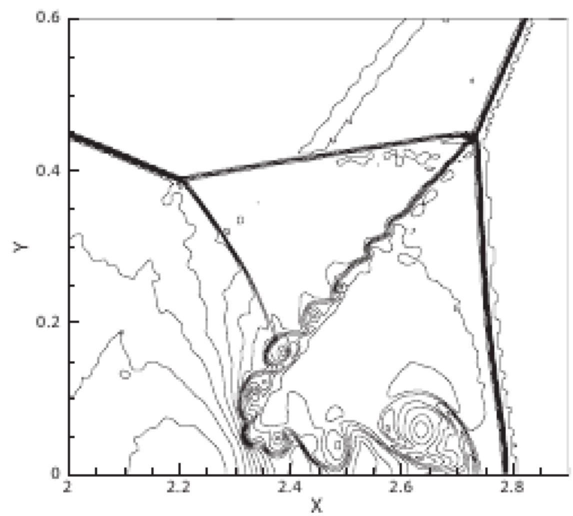
BVD-WENOZ-THINC



BVD-TENO5- THINC



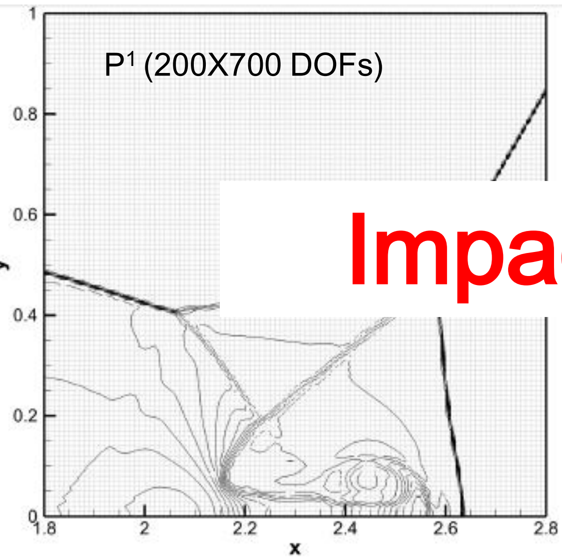
BVD-WENOZ-THINC-C

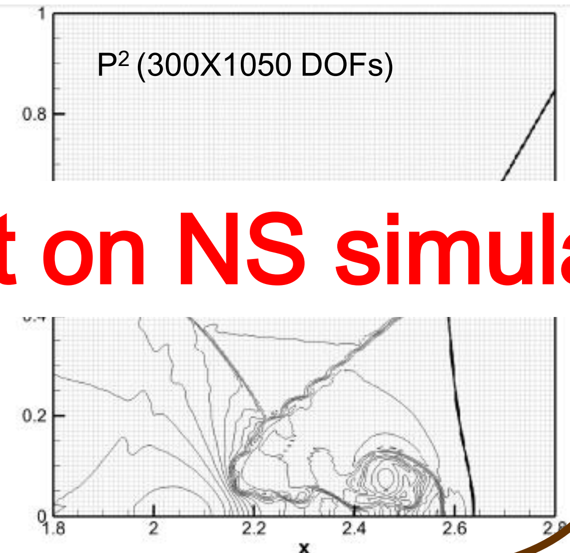


Comparison with high-order DG method

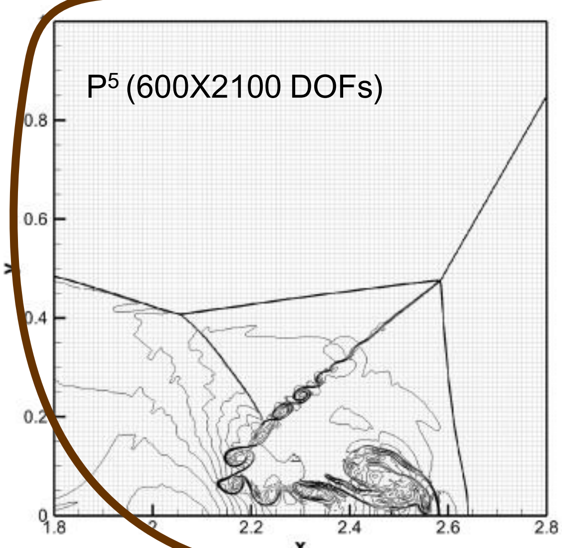
Dumbser et al, JCP 2014

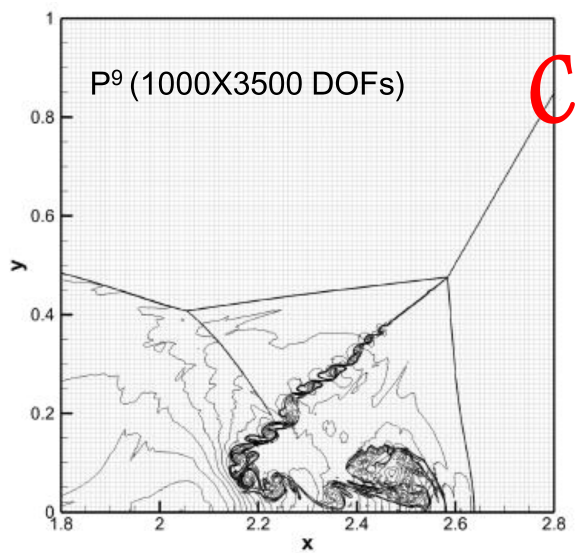
100X350 mesh with P^n DG reconstruction





Impact on NS simulations

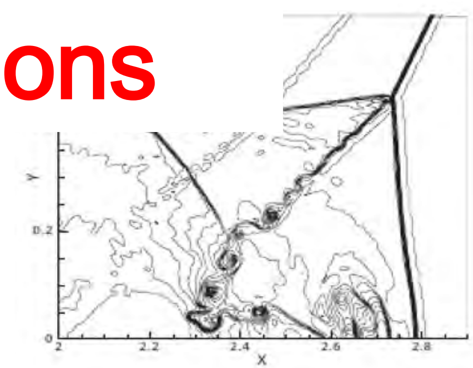




BVD reconstruction

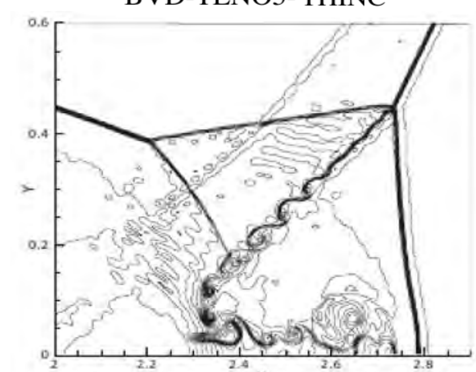
200X640

BVD-WENOZ-THINC



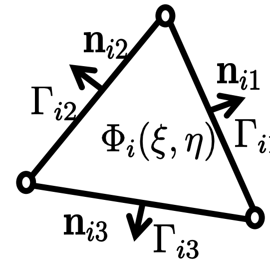
Comparable

BVD-TENO5- THINC



Implementing BVD principle on unstructured grids

BVD algorithm for unstructured grid (simplest version)



- Prepare $\Phi_i^{(1)}(x,y)$ and reconstruction $\Phi_i^{(2)}(x,y)$ for cell Ω_i.
- Evaluate TBV (total boundary variation) for both $\Phi_i^{(1)}(x,y)$ and $\Phi_i^{(2)}(x,y)$ by

$$TBV(\Phi)_i^{(1)} = \frac{\sum_{j=1}^{J}\left(\frac{1}{|\Gamma_{ij}|}\int_{\Gamma_{ij}}\Phi_i^{(1)}(x_{ij},y_{ij})\,d\Gamma - \frac{1}{|\Gamma_{ij}|}\int_{\Gamma_{ij}}\Phi_{ij}^{(1)}(x_{ij},y_{ij})\,d\Gamma\right)^{\theta}}{\sum_{j=1}^{J}\left(\overline{\phi}_i - \overline{\phi}_{ij}\right)^{\theta}},$$

$$TBV(\Phi)_i^{(2)} = \frac{\sum_{j=1}^{J}\left(\frac{1}{|\Gamma_{ij}|}\int_{\Gamma_{ij}}\Phi_i^{(2)}(x_{ij},y_{ij})\,d\Gamma - \frac{1}{|\Gamma_{ij}|}\int_{\Gamma_{ij}}\Phi_{ij}^{(1)}(x_{ij},y_{ij})\,d\Gamma\right)^{\theta}}{\sum_{j=1}^{J}\left(\overline{\phi}_i - \overline{\phi}_{ij}\right)^{\theta}}.$$

  Note that simplification is made by assuming the solution in neighboring cells are smooth and approximated by $\Phi_i^{(1)}$.

- The reconstruction $\Phi_i^{(3)}(x,y) = \omega_i\Phi_i^{(2)}(x,y) + (1-\omega_i)\Phi_i^{(1)}(x,y)$ where $\omega_i \in [0,1]$, a parameter weighting the reconstruction between $\Phi_i^{(1)}(x,y)$ and $\Phi_i^{(2)}(x,y)$.

Implementing BVD principle on unstructured grids

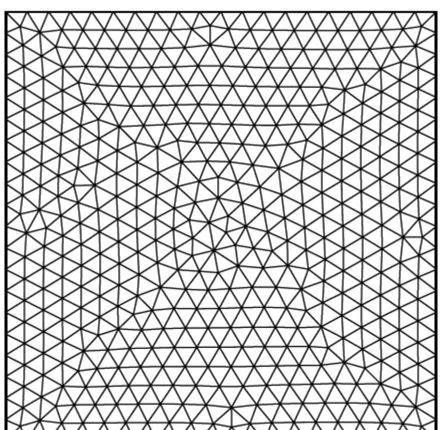
BVD algorithm for unstructured grid (simplest version)

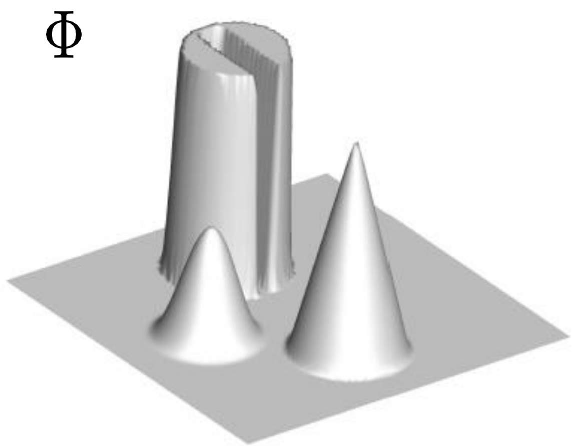
- Seeking a reconstruction function that minimize

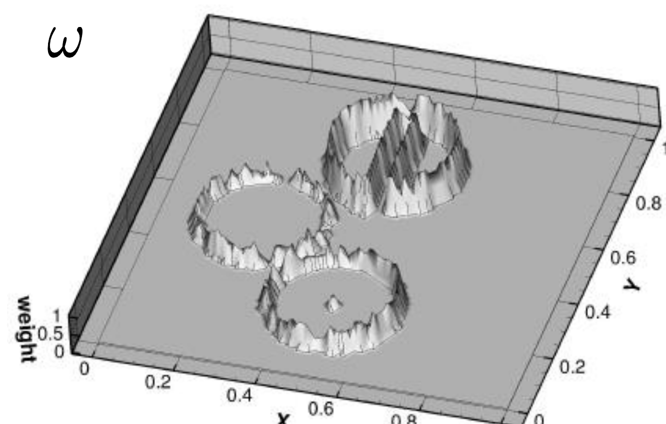
$$\epsilon_i = \sum_{j=1}^{J} \left(\int_{\Gamma_{ij}} \Phi_i^{(3)}(x_{ij}, y_{ij})\, d\Gamma - \int_{\Gamma_{ij}} \Phi_{ij}^{(1)}(x_{ij}, y_{ij})\, d\Gamma \right)^2$$

leads to a weight parameter ω_i computed from requirement

$$\frac{\partial \epsilon_i}{\partial \omega_i} = 0.$$

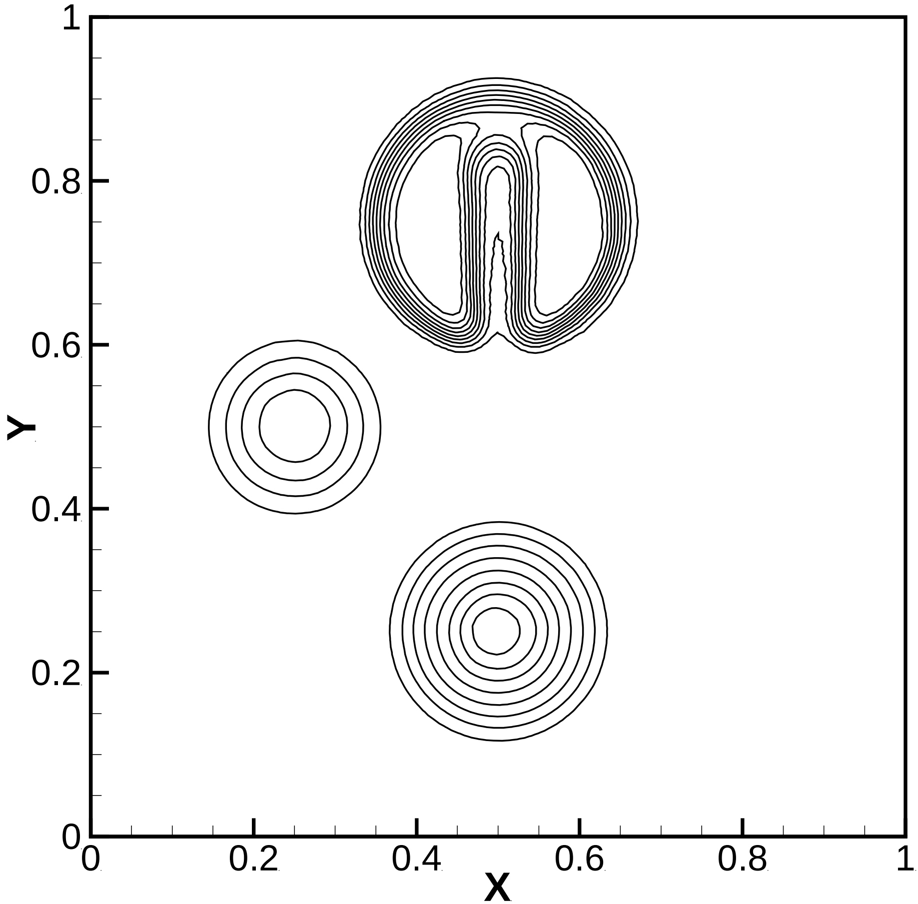




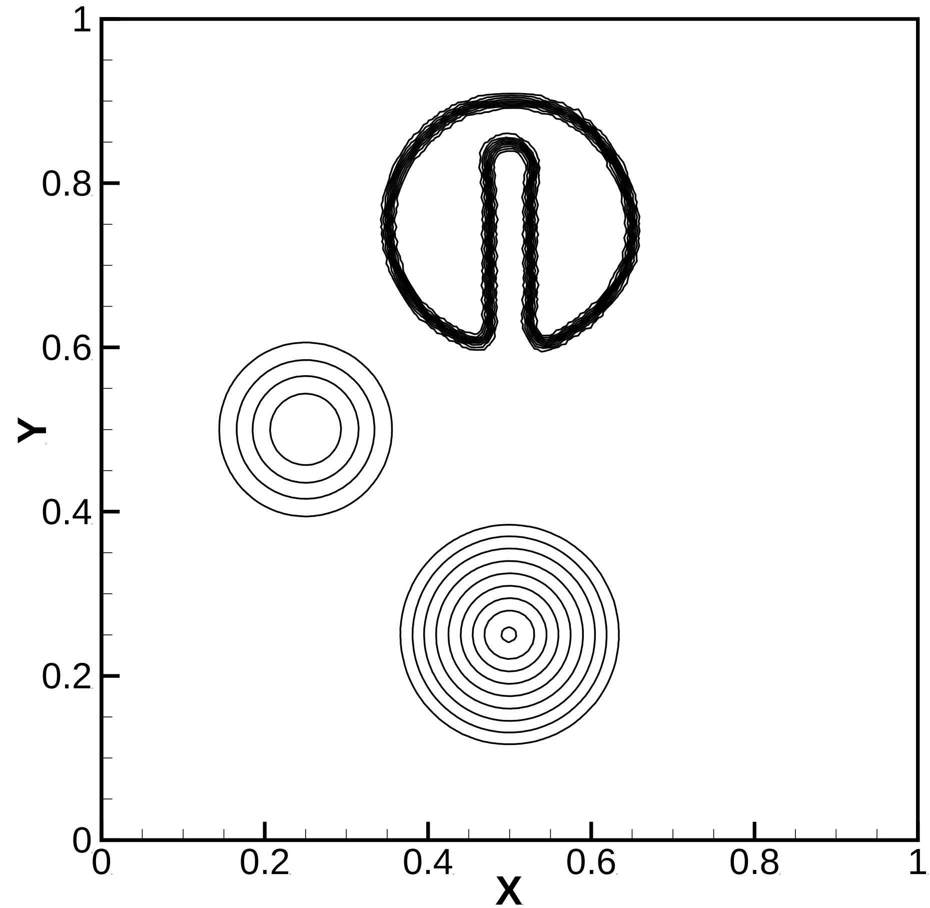


Advection test on unstructured grid

Triangular grid

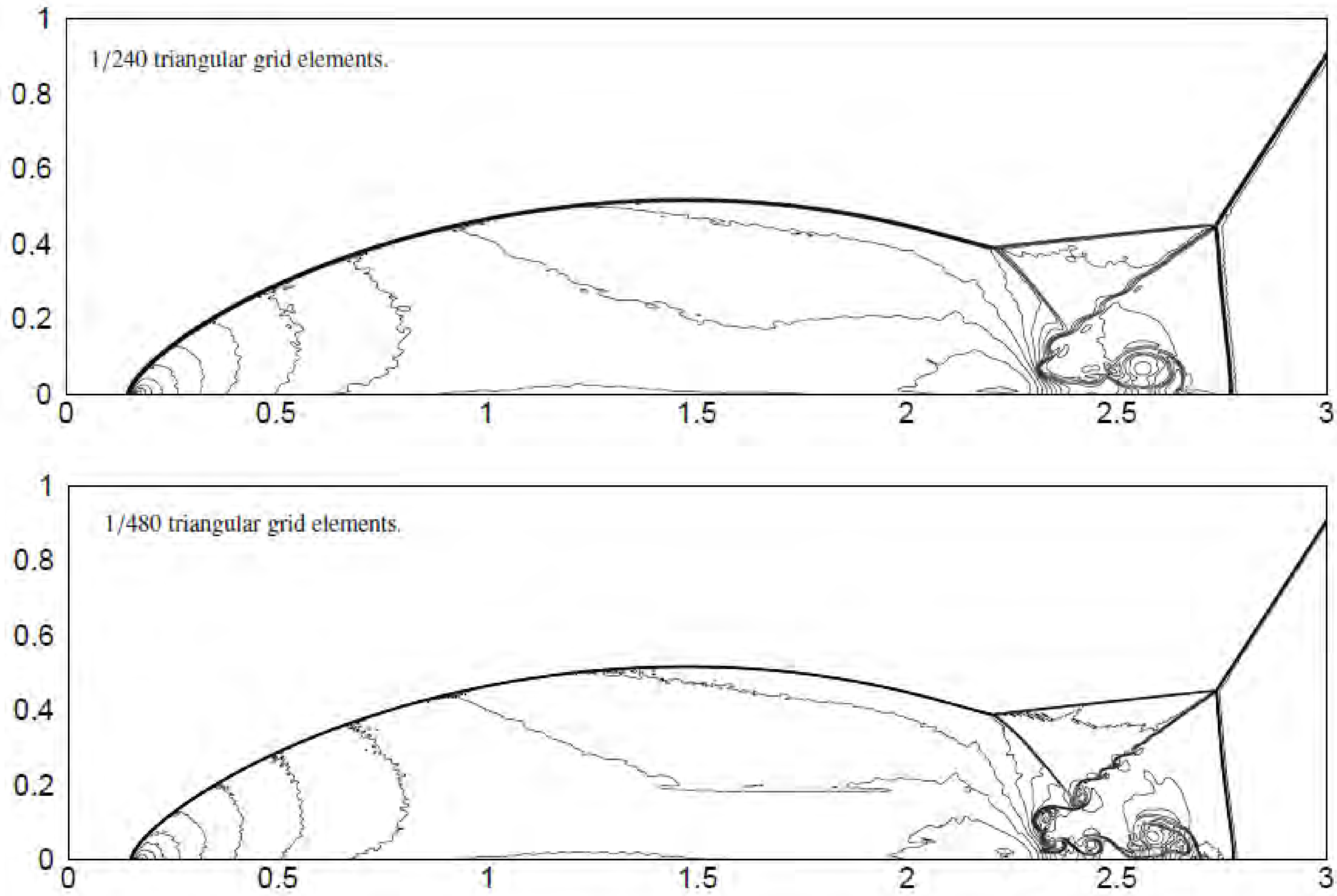


MUSCL



VPM-BVD

Double Mach reflection on unstructured grid



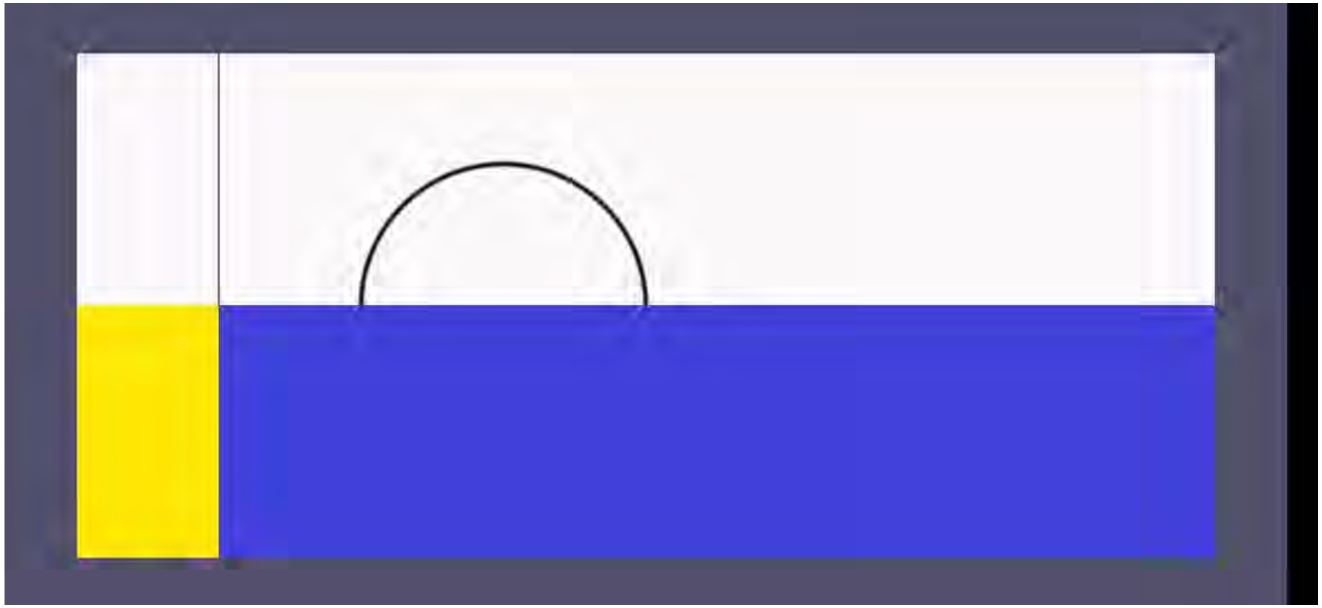
Compressible multiphase flow with interfaces

Air shock-R22 bubble *interaction*

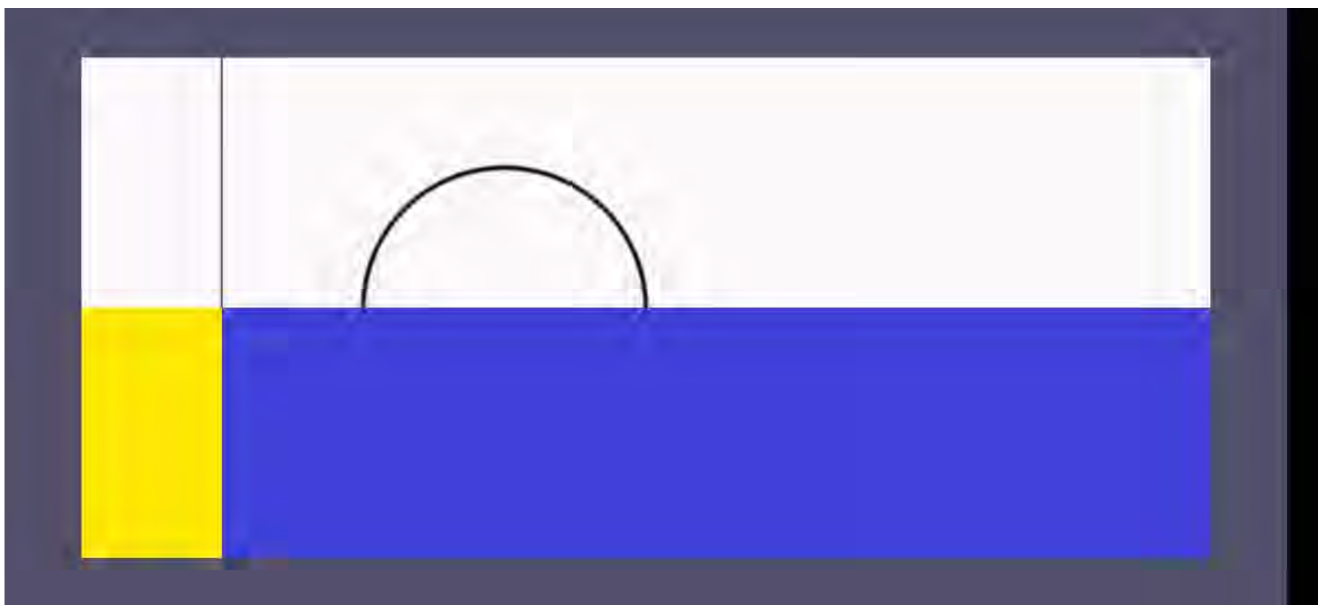
Hass,Sturtevant, J. Fluid Mech., 181, 41-76(1987)

Without BVD

Shyue & Xiao (2014) JCP ., 268, 326-354.



With BVD



Summary

- ✓ A new strategy to design high-fidelity schemes to capture both smooth and discontinuous solutions.
- ✓ A simple and accurate approach of great practical significance.
- ✓ Superior solution quality to other existing methods with the same DOFs.
- ✓ An approach that might lead to some new stories in related fields.

Thank you !